Planar Topology (continued)

Regions, Compactness, and Locally Uniform Convergence

Lecture Notes

October 8, 2026

1 Path-Connectedness (continued)

Theorem

Let $U \subseteq \mathbb{C}$ be open. Then U is connected if and only if it is path-connected.

Proof of the Theorem (open + connected $\implies$ path-connected). Fix $z \in U$ and let

$$U_z := \{w \in U : \text{there is a path in } U \text{ from } z \text{ to } w\}.$$

U_z is open: if $w \in U_z$, choose δ with $B(w,\delta) \subseteq U$; every $w' \in B(w,\delta)$ is joined to w by a straight segment in U, and w is joined to z by a path in U, so concatenating gives a path from z to w'; hence $B(w,\delta) \subseteq U_z$.

The same argument shows that $U \setminus U_z$ is a union of similarly-open "path components" of points not connected to z, so it is open. Since $U = U_z \cup (U \setminus U_z)$ is a disjoint union of open sets and U is connected with $U_z \ni z$ nonempty, we must have $U \setminus U_z = \varnothing$, i.e. $U = U_z$: every point of U is joined to z by a path in U. □

Remark. Even more is true, by the same proof: if U is open and connected, then any two points of U can be joined by a path consisting of finitely many segments each parallel to one of the coordinate axes (a "staircase path").

Theorem

If $U \subseteq \mathbb{C}$ is open, each of its connected components is open.

Proof. Write $U = \bigcup_i U_i$ (connected components). If $w \in U_i$, choose $B(w,\delta) \subseteq U$ (U open); $B(w,\delta)$ is connected and meets U_i (at w), so $U_i \cup B(w,\delta)$ is connected, forcing $B(w,\delta) \subseteq U_i$ by maximality of U_i. □

2 Regions and Analytic Functions

Definition (region)

A **region** is an open, connected subset of $\mathbb{C}$. We write $\mathcal{A}(D)$ for the functions analytic on a region D.

Theorem

Let D be a region, $f \in \mathcal{A}(D)$, $f' \equiv 0$. Then f is constant on D.

Proof. Fix $z_0 \in D$ and let $V := \{w \in D : f(w) = f(z_0)\}$. V is open: if $w \in V$, choose $B(w, \delta) \subseteq D$; since $f' \equiv 0$ on the disc, f is constant there (equal to $f(w) = f(z_0)$), so $B(w, \delta) \subseteq V$. By the same reasoning, $D \setminus V$ is open too: if $f(w) \neq f(z_0)$, then f is constant (equal to $f(w)$) on a disc around w, so that whole disc avoids V. Since $D = V \cup (D \setminus V)$ is a disjoint union of open sets, D is connected, and $V \ni z_0$ is nonempty, we conclude $D \setminus V = \varnothing$, i.e. $f \equiv f(z_0)$ on D. □

Definition ($\mathcal{A}(K)$ for general K)

f is **analytic on** $K \subseteq \mathbb{C}$ if there is an open $V \supseteq K$ and $F \in \mathcal{A}(V)$ with $F|_K = f$. Notation: $f \in \mathcal{A}(K)$.

3 Compactness

Facts about compact sets

Let K be a closed, bounded ($\exists R : K \subseteq B(0, R)$) subset of $\mathbb{C}$. Then:

(1) (**Compactness**) If $K \subseteq \bigcup_{i \in I} U_i$ with U_i open, then $K \subseteq U_1 \cup \cdots \cup U_n$ for finitely many of the U_i.

(2) (**Sequential compactness**) Every sequence $(z_n) \subseteq K$ has a subsequence $z_{n_k} \to z \in K$.

(3) If $f : K \to \mathbb{R}$ is continuous, $f(K)$ is closed and bounded; in particular $\max_K f$ and $\min_K f$ are finite and attained.

(4) If $f : K \to \mathbb{C}$ is continuous, $f(K)$ is closed and bounded.

(5) If $f : K \to \mathbb{R}$ or $f : K \to \mathbb{C}$ is continuous, f is **uniformly continuous**: $\forall \varepsilon > 0\ \exists \delta > 0 : |z_1 - z_2| < \delta \implies |f(z_1) - f(z_2)| < \varepsilon$.

(These are standard facts from the Heine–Borel theorem; no proof given here.)

Remark. Every closed subset of $\widehat{\mathbb{C}}$ is compact. For $K \subseteq \mathbb{C}$ closed: K is closed in $\widehat{\mathbb{C}} \iff K$ is bounded (an unbounded closed K has ∞ in its closure in $\widehat{\mathbb{C}}$, but $\infty \notin K$).

Theorem (nested compact sets)

If (K_n) is a decreasing sequence of nonempty compact sets, then $\bigcap_n K_n \neq \varnothing$.

Proof. If $\bigcap_n K_n = \varnothing$, let $U_n := K_n^c$ (open, relative to K_1). Then $K_1 \subseteq \bigcup_n U_n$, so by compactness $K_1 \subseteq U_1 \cup \cdots \cup U_n = U_n$ (as the U_n are nested increasing), i.e. $K_1 \cap K_n = K_n = \varnothing$, contradicting $K_n \neq \varnothing$. □

Remark. The same holds for any closed $K \subseteq \widehat{\mathbb{C}}$, using the spherical metric.

4 An Application of Compactness: Locally Uniform Convergence

Definition (locally uniform convergence)

Let $S \subseteq \mathbb{C}$, $f_n, f : S \to \mathbb{C}$. We say $f_n \to f$ **locally uniformly** if

$$\forall \varepsilon > 0 \; \forall z \in S \; \exists \delta(\varepsilon, z) > 0, \; N(\varepsilon, z) : n > N, \; w \in B(z, \delta) \cap S \implies |f_n(w) - f(w)| \leq \varepsilon.$$

Since δ and N are allowed to depend on both ε *and* z, this is a much weaker requirement than uniform convergence.

Theorem

If S is compact and $f_n \to f$ locally uniformly, then $f_n \to f$ uniformly on S.

Proof. Fix $\varepsilon > 0$. Then $S \subseteq \bigcup_{z \in S} B(z, \delta(\varepsilon, z))$; by compactness, $S \subseteq B(z_1, \delta_1) \cup \cdots \cup B(z_k, \delta_k)$ for finitely many $z_1, \ldots, z_k$. Let $N := \max_{1 \leq j \leq k} N(\varepsilon, z_j)$. Given $w \in S$, choose j with $w \in B(z_j, \delta_j)$; then $n > N \implies n > N(\varepsilon, z_j) \implies |f_n(w) - f(w)| < \varepsilon$. □

Corollary

Let $S \subseteq \mathbb{C}$ be open, $f_n \to f$ locally uniformly, with each f_n continuous. Then f is continuous.

Proof. Fix $z \in S$ and δ with $\overline{B(z, \delta)} \subseteq S$; this closed disc is compact, so $f_n \to f$ uniformly on it, hence f is continuous there (a uniform limit of continuous functions is continuous), in particular at z. □